B.Sc. Part I 1st paper

TRIGONOMETRY Summation of series (contd.)

Q. Sum the following series

$$k\cos\alpha - \frac{k^2}{2}\cos(\alpha+\beta) + \frac{k^3}{3}\cos(\alpha+2\beta) \dots \text{to } \infty$$

Soln. Let the given series be denoted by C.

$$\Rightarrow C = k\cos\alpha - \frac{k^2}{2}\cos(\alpha+\beta) + \frac{k^3}{3}\cos(\alpha+2\beta) \dots \text{to } \infty$$

Let $$S = k\sin\alpha - \frac{k^2}{2}\sin(\alpha+\beta) + \frac{k^3}{3}\sin(\alpha+2\beta) \dots \text{to } \infty$$

$$\Rightarrow C+iS = k(\cos\alpha + i\sin\alpha) - \frac{k^2}{2}\left[\cos(\alpha+\beta)+i\sin(\alpha+\beta)\right] + \frac{k^3}{3}\left[\cos(\alpha+2\beta)+i\sin(\alpha+2\beta)\right] \dots \text{to } \infty$$

$$\Rightarrow C+iS = ke^{i\alpha} - \frac{k^2}{2}e^{i(\alpha+\beta)} + \frac{k^3}{3}e^{i(\alpha+2\beta)} \dots \text{to } \infty$$

$$\Rightarrow C+iS = ke^{i\alpha} - \frac{k^2}{2}\cdot e^{i\alpha}\cdot e^{i\beta} + \frac{k^3}{3}e^{i\alpha}\cdot e^{2i\beta} \dots \text{to } \infty$$

$$\Rightarrow C+iS = e^{i\alpha}\left[k - \frac{k^2}{2}e^{i\beta} + \frac{k^3}{3}e^{2i\beta} \dots \text{to } \infty\right]$$

$$= \frac{e^{i\alpha}}{e^{i\beta}}\cdot e^{i\beta}\left[k - \frac{k^2}{2}e^{i\beta} + \frac{k^2}{3}e^{2i\beta} \dots \text{to } \infty\right]$$

$$\Rightarrow C+iS = e^{i(\alpha-\beta)}\left[ke^{i\beta} - \frac{k^2}{2}e^{2i\beta} + \frac{k^3}{3}e^{3i\beta} - \cdots \text{to}\infty\right]$$

Put $ke^{i\beta} = x$

$$\Rightarrow C+iS = e^{i(\alpha-\beta)}\left[x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \text{to}\infty\right]$$

$$\Rightarrow C+iS = e^{i(\alpha-\beta)} \log(1+x)$$

$$= e^{i(\alpha-\beta)} \log\{1+ke^{i\beta}\}$$

$$\Rightarrow C+iS = e^{i(\alpha-\beta)} \log(1+k\cos\beta + ik\sin\beta)$$

$$\Rightarrow C+iS = e^{i(\alpha-\beta)}\left[\frac{1}{2}\log\{(1+k\cos\beta)^2 + (k\sin\beta)^2\} \cancel{+ i\tan} + i\tan^{-1}\frac{k\sin\beta}{1+k\cos\beta}\right]$$

$$\Rightarrow C+iS = \left[\cos(\alpha-\beta) + i\sin(\alpha-\beta)\right]\left[\frac{1}{2}\log(1+2k\cos\beta+k^2) + i\tan^{-1}\frac{k\sin\beta}{1+k\cos\beta}\right]$$

Equating real parts, we get

$$\Rightarrow C = \frac{1}{2}\cos(\alpha-\beta)\log(1+2k\cos\beta+k^2) \cancel{- \sin(\alpha-\beta)\tan^{-1}\ldots} - \sin(\alpha-\beta)\tan^{-1}\frac{k\sin\beta}{1+k\cos\beta}$$